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Spinor parallel propagator and Green function in maximally symmetric spaces

Wolfgang Mück
Department of Physics, Simon Fraser University, Burnaby, BC, Canada V5A 1S6
E-mail: wmueck@sfu.ca

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Abstract. We introduce the spinor parallel propagator for maximally symmetric spaces in any dimension. Then, the Dirac spinor Green functions in the maximally symmetric spaces $\mathbb{R}^n$, S^n and H^n are calculated in terms of intrinsic geometric objects. The results are covariant and coordinate independent.

1. Introduction

The study of field theory in anti-de Sitter (AdS) spaces, which topologically are hyperbolic maximally symmetric spaces, has been revived over the past two years following the so-called Maldacena conjecture relating type IIB supergravity on $AdS_5 \times S^5$ with $\mathcal{N} = 4$, $U(N)$ super-Yang–Mills theory in four dimensions [1].

More than a decade ago, the calculation of correlation functions in maximally symmetric spaces using only intrinsic geometric objects was presented in a series of papers starting with [2–4]. In one of them [4], Green functions for two-component spinors in maximally symmetric 4-spaces were considered using the $SL(2, \mathbb{R})$ formulation. To our knowledge, this analysis has not been extended to Dirac spinors in other spacetime dimensions. However, it should be mentioned that spinor Green functions in AdS spaces have recently been considered and calculated by other means in the context of the AdS–CFT correspondence [5, 6].

Here, we present an intrinsically geometric approach to spinor Green functions in maximally symmetric spaces. In section 2, we introduce the spinor parallel propagator for maximally symmetric spaces of dimension n and find its covariant derivatives. Then, in section 3, we calculate the spinor Green functions for the spaces $\mathbb{R}^n$, S^n and H^n. Finally, section 4 contains conclusions.

In the remainder of this section, we would like to review the elementary maximally symmetric bi-tensors, which have been discussed in detail by Allen and Jacobson [3].

Consider a maximally symmetric space of dimension n with constant scalar curvature $n(n-1)/R^2$. For the space S^n, the radius R is real and positive, whereas for the hyperbolic space H^n, $R = \mathrm{i}l$ with l positive, and in the flat case, $\mathbb{R}^n$, $R = \infty$.

Consider further two points x and x', which can be connected uniquely by a shortest geodesic. Let μ be the proper geodesic distance along this shortest geodesic between x and x'. We shall denote the covariant derivatives with respect to x and x' by D_μ and $D_{\mu'}$, respectively. Then, the vectors

$$n_\nu(x, x') = D_\nu \mu(x, x') \qquad \text{and} \qquad n_{\nu'}(x, x') = D_{\nu'} \mu(x, x') \tag{1}$$

are tangent to the geodesic and have unit length. Furthermore, denote by $g^{\mu}{}_{\nu'}(x, x')$ the vector parallel propagator along the geodesic. Note the relation $n^{\nu'} = -g^{\nu'}{}_{\mu} n^{\mu}$.

These elementary maximally bi-tensors n^{μ}, $n^{\mu'}$ and $g^{\mu}{}_{\nu'}$ satisfy the following properties:

$$D_\mu n_\nu = A(g_{\mu\nu} - n_\mu n_\nu) \tag{2a}$$

$$D_{\mu'} n_\nu = C(g_{\mu'\nu} + n_{\mu'} n_\nu) \tag{2b}$$

$$D_\mu g_{\nu\lambda'} = -(A + C)(g_{\mu\nu} n_{\lambda'} + g_{\mu\lambda'} n_\nu) \tag{2c}$$

where A and C are functions of the geodesic distance μ and are given by

$$A = \frac{1}{R} \cot \frac{\mu}{R} \qquad \text{and} \qquad C = -\frac{1}{R \sin(\mu/R)}. \tag{3}$$

Therefore, they satisfy the relations

$$\mathrm{d}A/\mathrm{d}\mu = -C^2 \qquad \mathrm{d}C/\mathrm{d}\mu = -AC \qquad \text{and} \qquad C^2 - A^2 = 1/R^2. \tag{4}$$

The covariant derivative of spinors is defined in the usual sense using spin connections,

$$[D_\mu \psi(x)]^\alpha = \partial_\mu \psi^\alpha(x) + \tfrac{1}{2} \omega_\mu{}^{ab} [S_{ab} \psi(x)]^\alpha \tag{5}$$

where $S_{ab} = \frac{1}{4}[\gamma_a, \gamma_b]$ are the usual spin matrices, and γ_a are the Dirac gamma matrices of the local Lorentz frame, $\{\gamma_a, \gamma_b\} = 2\eta_{ab}$. The covariant gamma matrices are defined by $\Gamma^\mu(x) = e^\mu_a(x)\gamma^a$, where $e^\mu_a(x)$ is a vielbein. Then, the covariant gamma matrices satisfy $\{\Gamma^\mu, \Gamma^\nu\} = 2g^{\mu\nu}$. Equation (5) was given only for completeness, as we shall only need the covariant expressions.

2. Spinor parallel propagator

To start, consider a bi-spinor $\Lambda(x', x)^{\alpha'}{}_\beta$, which acts as a parallel propagator for Dirac spinors in a maximally symmetric spacetime, i.e. it performs the parallel transport

$$\Psi'(x')^{\alpha'} = \Lambda(x', x)^{\alpha'}{}_\beta \Psi(x)^\beta.$$

The spinor parallel propagator $\Lambda(x', x)$ can be uniquely defined for any spacetime dimension by the following properties:

$$\Lambda(x', x) = [\Lambda(x, x')]^{-1} \tag{6a}$$

$$\Gamma^{\nu'}(x') = \Lambda(x', x)\Gamma^\mu(x)\Lambda(x, x') g^{\nu'}{}_\mu(x', x) \tag{6b}$$

$$n^\mu D_\mu \Lambda(x, x') = 0. \tag{6c}$$

Equation (6*a*) implies that $\Lambda(x, x)^{\alpha'}{}_\beta = \delta^{\alpha'}_\beta$, whereas (6*b*) conveniently formulates the parallel transport of the covariant gamma matrices. Finally, equation (6*c*) says that $\Lambda(x, x')$ is covariantly constant along the geodesic of parallel transport.

We would now like to evaluate a particular property of $\Lambda(x, x')$, namely its covariant derivative. Therefore, combine equations (6*a*) and (6*b*) to give

$$\Gamma^\nu \Lambda(x, x') = \Lambda(x, x')\Gamma^{\mu'} g^\nu{}_{\mu'} \tag{7}$$

and differentiate covariantly with respect to x to obtain

$$\Gamma^\nu D_\lambda \Lambda(x, x') = D_\lambda \Lambda(x, x')\Gamma^{\mu'} g^\nu{}_{\mu'} - (A + C)\Lambda(x, x')\Gamma^{\mu'}(\delta^\nu_\lambda n_{\mu'} + g_{\lambda\mu'} n^\nu) \tag{8}$$

where we have used the property (2*c*) of the vector parallel propagator. Now, use (7) for the second term on the right-hand side of (8) and multiply by Γ^λ, which yields

$$2D^\nu \Lambda(x, x') - \Gamma^\nu \not{D}\Lambda(x, x') = \not{D}\Lambda(x, x')\Gamma^{\mu'} g^\nu{}_{\mu'} + (A + C)(\Gamma^\nu \Gamma^\rho n_\rho - n n^\nu)\Lambda(x, x'). \tag{9}$$

Thus, multiplication by Γ_ν leads to

$$(2-n)\not{D}\Lambda(x,x') = \Gamma_\nu \not{D}\Lambda(x,x')\Gamma^{\mu'} g^{\nu}{}_{\mu'}$$

the solution of which is

$$\not{D}\Lambda(x,x') = Bn_\mu \Gamma^\mu \Lambda(x,x') \tag{10}$$

where B is some function of the geodesic distance μ. Then, substituting (10) into (9) yields

$$2D^\nu \Lambda(x,x') = 2Bn^\nu \Lambda(x,x') + (A+C)(\Gamma^\nu \Gamma^\rho n_\rho - nn^\nu)\Lambda(x,x').$$

Moreover, by multiplying this by n_ν and using (6*c*), one determines B to be

$$B = \tfrac{1}{2}(n-1)(A+C).$$

Therefore, finally one obtains

$$D_\mu \Lambda(x,x') = \tfrac{1}{2}(A+C)\left(\Gamma_\mu \Gamma^\nu n_\nu - n_\mu\right)\Lambda(x,x'). \tag{11}$$

For completeness, we also give the expression for $D_{\mu'}\Lambda(x,x')$. It is easily obtained from (11) using (6*a*) and is given by

$$D_{\mu'}\Lambda(x,x') = -\tfrac{1}{2}(A+C)\Lambda(x,x')\left(\Gamma_{\mu'}\Gamma^{\nu'} n_{\nu'} - n_{\mu'}\right). \tag{12}$$

3. Spinor Green function

Using the spinor parallel propagator $\Lambda(x,x')$ introduced in section 2, we would now like to find the spinor Green function $S(x,x')$ satisfying

$$[(\not{D}-m)S(x,x')]^\alpha{}_{\beta'} = \frac{\delta(x-x')}{\sqrt{g(x)}}\delta^\alpha_{\beta'}. \tag{13}$$

Here, we have written the indices explicitly in order to emphasize that this is a bi-spinor equation. Henceforth we shall omit the indices.

Now, we make the general ansatz

$$S(x,x') = \left[\alpha(\mu) + \beta(\mu) n_\nu \Gamma^\nu\right]\Lambda(x,x'). \tag{14}$$

We substitute the ansatz (14) into (13) and, after using (11), obtain the two coupled differential equations

$$\beta' + \tfrac{1}{2}(n-1)(A-C)\beta - m\alpha = \frac{\delta(x-x')}{\sqrt{g(x)}} \tag{15}$$

$$\alpha' + \tfrac{1}{2}(n-1)(A+C)\alpha - m\beta = 0 \tag{16}$$

where the prime denotes differentiation with respect to μ.

In order to proceed, multiply (15) by m and substitute $m\beta$ from (16). One finds

$$\alpha'' + (n-1)A\alpha' - \tfrac{1}{2}(n-1)C(A+C)\alpha - \left[\frac{(n-1)^2}{4R^2} + m^2\right]\alpha = m\frac{\delta(x-x')}{\sqrt{g(x)}} \tag{17}$$

where (4) has been used. We shall solve (17) separately for the spaces $\mathbb{R}^n$, S^n and H^n.

3.1. Green function for $\mathbb{R}^n$

For $\mathbb{R}^n$, we have $A = -C = 1/\mu$, $R = \infty$ and $\mu = |x - x'|$. Thus, equation (17) becomes

$$\alpha'' + \frac{n-1}{\mu}\alpha' - m^2\alpha = m\,\delta(x - x'). \tag{18}$$

The solution of (18) is

$$\alpha(\mu) = -\left(\frac{m}{2\pi}\right)^{n/2} \mu^{1-n/2}\,\mathrm{K}_{n/2-1}(m\mu) \tag{19}$$

where the functional form was obtained by solving (18) for $\mu \neq 0$, and the constant was found by matching the singularity. Furthermore, one finds from (16) $m\beta = \alpha'$, i.e. $n_\nu\beta = \partial_\nu\alpha/m$, so that the final result for the spinor Green function in $\mathbb{R}^n$ is

$$S(x, x') = -\frac{1}{m}\left(\frac{m}{2\pi}\right)^{n/2} (\not{\partial} + m)\mu^{1-n/2}\,\mathrm{K}_{n/2-1}(m\mu). \tag{20}$$

Upon Fourier transforming it, one obtains the more familiar expression

$$S(x, x') = -(\not{\partial} + m)\int \frac{\mathrm{d}^n k}{(2\pi)^n}\,\mathrm{e}^{-\mathrm{i}k(x-x')}\frac{1}{k^2 + m^2}. \tag{21}$$

3.2. Green function for S^n

In order to solve (17), we first consider $x \neq x'$ and make the substitution

$$z = \cos^2\frac{\mu}{2R}. \tag{22}$$

This yields the differential equation

$$\left[z(1-z)\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \frac{n}{2}(1-2z)\frac{\mathrm{d}}{\mathrm{d}z} - \frac{(n-1)^2}{4} - m^2R^2 - \frac{n-1}{4z}\right]\alpha(z) = 0. \tag{23}$$

Then, writing $\alpha(z) = \sqrt{z}\,\gamma(z)$, one obtains a hypergeometric equation for γ,

$$H(a, b; c; z)\gamma(z) = 0 \tag{24a}$$

where

$$H(a, b; c; z) = z(1-z)\frac{\mathrm{d}^2}{\mathrm{d}z^2} + [c - (a+b+1)z]\frac{\mathrm{d}}{\mathrm{d}z} - ab \tag{24b}$$

is the hypergeometric operator, and its parameters are

$$a = \tfrac{1}{2}n - \mathrm{i}|m|R \qquad b = \tfrac{1}{2}n + \mathrm{i}|m|R \qquad c = \tfrac{1}{2}n + 1. \tag{24c}$$

The solution of (24) that is singular at $z = 1$ is [7]

$$\gamma(z) = \lambda\,F(a, b; c; z) = \lambda\,F(n/2 - \mathrm{i}|m|R, n/2 + \mathrm{i}|m|R; n/2 + 1; z) \tag{25}$$

where λ is a proportionality constant. Therefore, $\alpha(z)$ is

$$\alpha(z) = \lambda\sqrt{z}\,F(n/2 - \mathrm{i}|m|R, n/2 + \mathrm{i}|m|R; n/2 + 1; z). \tag{26}$$

We can now determine the constant λ by matching the singularity in (17). This is equivalent to demanding the singularity of α at $\mu = 0$ to have the same strength as in the case of $\mathbb{R}^n$. One finds from (26)

$$\alpha \to \lambda\frac{\Gamma(n/2+1)\Gamma(n/2-1)}{\Gamma(n/2 - \mathrm{i}|m|R)\Gamma(n/2 + \mathrm{i}|m|R)}\left(\frac{\mu}{2R}\right)^{2-n}$$

whereas in $\mathbb{R}^n$ we have, from (19),

$$\alpha \to -\tfrac{1}{4}m\Gamma(n/2-1)\pi^{-n/2}\mu^{2-n}. \tag{27}$$

Comparing these two expressions we find

$$\lambda = -m\frac{\Gamma(n/2-\mathrm{i}|m|R)\Gamma(n/2+\mathrm{i}|m|R)}{\Gamma(n/2+1)\pi^{n/2}2^n}R^{2-n}. \tag{28}$$

Finally, one can calculate β from (16), which yields

$$\begin{aligned}\beta(z) &= -\frac{1}{m}\left[\frac{1}{R}\sqrt{z(1-z)}\frac{\mathrm{d}}{\mathrm{d}z}+\frac{n-1}{2R}\sqrt{\frac{1-z}{z}}\right]\alpha(z)\\ &= -\frac{\lambda}{mR}\sqrt{1-z}\big[z\,F(n/2+1-\mathrm{i}|m|R,n/2+1+\mathrm{i}|m|R;n/2+2;z)\\ &\quad+\tfrac{1}{2}n\,F(n/2-\mathrm{i}|m|R,n/2+\mathrm{i}|m|R;n/2+1;z)\big].\end{aligned} \tag{29}$$

It should be noted that β has a finite $m \to 0$ limit, whereas α vanishes.

3.3. Green function for H^n

For H^n, we can start with (24) and set $R = \mathrm{i}l$, i.e. we have to solve

$$H(a,b;c;z)\gamma(z) = 0 \tag{30a}$$

with

$$a = \tfrac{1}{2}n+|m|l \qquad b = \tfrac{1}{2}n-|m|l \qquad c = \tfrac{1}{2}n+1. \tag{30b}$$

There are two solutions to (30) which behave asymptotically like a power of z for $z \to \infty$. These are

$$\gamma_\pm(z) = \lambda_\pm z^{-(n/2\pm|m|l)}\,F\left(\tfrac{1}{2}n\pm|m|l,\pm|m|l;1\pm 2|m|l;\frac{1}{z}\right) \tag{31}$$

where $\lambda_\pm$ are constants. The choice of the minus sign is not always possible. In fact, for $1-2|m|l = 0,-1,-2,\ldots$ the hypergeometric series is indeterminate. Thus, we shall include the solution with the minus sign only, if $|m|l < \frac{1}{2}$. Hence, we have two solutions for α,

$$\alpha_\pm(z) = \lambda_\pm z^{-[(n-1)/2\pm|m|l]}\,F\left(\tfrac{1}{2}n\pm|m|l,\pm|m|l;1\pm 2|m|l;\frac{1}{z}\right) \tag{32}$$

and we can now proceed to determine the constants $\lambda_\pm$ in a similar fashion to the S^n case. From (32) we find for $\mu \to 0$

$$\alpha \to \lambda_\pm\left(\frac{\mu}{2l}\right)^{2-n}\frac{\Gamma(1\pm 2|m|l)\Gamma(n/2-1)}{\Gamma(n/2\pm|m|l)\Gamma(\pm|m|l)}.$$

Comparing this expression with the $\mathbb{R}^n$ case, equation (27), we find

$$\lambda_\pm = \mp\,\mathrm{sgn}\,m\,2^{-[n\pm 2|m|l]}l^{1-n}\frac{\Gamma(n/2\pm|m|l)}{\pi^{(n-1)/2}\Gamma\left(\frac{1}{2}\pm|m|l\right)} \tag{33}$$

where the doubling formula for gamma functions has been used.

Finally, let us calculate β from (16). Using a recursion formula for hypergeometric functions we find

$$\beta_\pm(z) = \frac{1}{m}\left[\frac{1}{l}\sqrt{z(z-1)}\frac{\mathrm{d}}{\mathrm{d}z} + \frac{n-1}{2l}\sqrt{\frac{z-1}{z}}\right]\alpha_\pm(z)$$
$$= \mp\,\mathrm{sgn}\,m\,\lambda_\pm\sqrt{z-1}\,z^{-(n/2\pm|m|l)}\,F\left(\tfrac{1}{2}n \pm |m|l, 1\pm |m|l; 1\pm 2|m|l; \frac{1}{z}\right). \quad (34)$$

It is interesting to note that in the limit $m \to 0$ the functions β_+ and β_- become identical, whereas α_+ and α_- do not, but differ in their signs. The reason is, of course, that, for $m = 0$, equations (15) and (16) decouple, and α can be a solution of (16) with an arbitrary proportionality constant. Moreover, for $m = 0$, the common value of $\beta_\pm$ is a rational function of z,

$$\beta_\pm(z) = \frac{\Gamma(n/2)}{(2\pi)^n} l^{1-n}(z-1)^{-(n-1)/2}.$$

4. Conclusions

We have introduced the spinor parallel propagator for maximally symmetric spaces in any dimension. This enabled us to find expressions for the Dirac spinor Green functions in the maximally symmetric spaces $\mathbb{R}^n$, S^n and H^n in terms of intrinsic geometric objects. Although there are obstructions to the quantization of spinors in odd-dimensional manifolds with a boundary [8], our results should be applicable to the AdS–CFT correspondence, because of the classical dynamics in AdS space.

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